A new critical curve for a class of quasilinear elliptic systems

Lorenzo D'Ambrosio
Dipartimento di Matematica, Università degli Studi di Bari
via E. Orabona, 4, I-70125 Bari, Italy, dambros@dm.uniba.it

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Abstract

We study a class of systems of quasilinear differential inequalities associated to weakly coercive differential operators and power reaction terms. The main model cases are given by the p-Laplacian operator as well as the mean curvature operator in non parametric form. We prove that if the exponents lie under a certain curve, then the system has only the trivial solution. These results hold without any restriction provided the possible solutions are more regular. The underlying framework is the classical Euclidean case as well as the Carnot groups setting.

Keywords Quasilinear elliptic systems, a priori estimates, Liouville theorems, Carnot groups.
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1 Introduction

Liouville theorems for elliptic systems is a classical subject in the theory of partial differential equations. In recent years, many general results have been obtained by several authors.

In this paper we shall consider a canonical class of elliptic systems of Hamiltonian

type of the form,

$$
\begin{cases}
L_1(u) := \mathrm{div}\mathscr{A}_1(x,u,\nabla u) = H_v(x,u,v) \quad on \quad \mathbb{R}^N, \\
\\
L_2(v) := \mathrm{div}\mathscr{A}_2(x,v,\nabla v) = H_u(x,u,v) \quad on \quad \mathbb{R}^N, \\
\\
u \geq 0, \quad v \geq 0 \quad on \quad \mathbb{R}^N.
\end{cases} \tag{1}
$$

Here $L_i, i = 1, 2$, are quasilinear elliptic operators satisfying a weak coercivity assumption (see below for the precise assumption) and $H : \mathbb{R}^2 \to \mathbb{R}$ is a given function controlled from below by a positive polynomial in the variables (u, v). Notice that if in (1), at least one of the operators L_i is quasilinear, the problem is not variational. Plainly the same holds when both operators L_i are linear but $L_1 \neq L_2^*$ (here L_2^* denotes the formal adjoint of L_2).

As it is well known one of the central themes for proving Liouville theorems for problem (1) is to find good (possibly sharp) a priori estimates on the weak solutions. These estimates are very difficult to prove under the weak coercivity assumption of $L_i, i = 1, 2$ (see the next section for the precise definition). This is due mainly to the lack of a weak Harnack's inequality.

Results on a priori estimates of weak solutions for non coercive quasilinear elliptic systems were first obtained by Mitidieri and Pohozaev in [26], [27] and by Bidaut-Véron and Pohozaev in [2] under weak structure assumptions on the differential operators. In those works, particular important examples of systems involving the p-Laplacian operator were considered.

Very recently these problems have received a renewed interest for coercive systems. See for instance Bidaut-Véron, Garcia-Huidobro and Yarur [1] for a recent interesting contribution in this direction.

In this paper we shall focus our attention on weakly coercive elliptic systems of inequalities of the canonical form,

$$
\begin{cases}
\mathrm{div}\mathscr{A}_1(x,u,\nabla u) \geq v^p \quad on \quad \mathbb{R}^N, \\
\\
\mathrm{div}\mathscr{A}_2(x,v,\nabla v) \geq u^q \quad on \quad \mathbb{R}^N, \\
\\
u \geq 0, \quad v \geq 0 \quad on \quad \mathbb{R}^N.
\end{cases} \tag{2}
$$

Many authors studied system related to (2). We refer the interested reader to Yarur [39], Lair and Wood [21], Teramoto [35], Cîrstea and Rădulescu [7], Yang [38] and the references therein. The systems studied by these authors contain nonautonomous nonlinearities and the differential operators are the Laplace operator or the p-Laplacian operator. Moreover,

many of these papers deal with radial solutions of systems of equations. Our results hold even for some classes of nonautonomous systems without assuming the radial symmetry of the possible solutions. Throughout this paper, in order to avoid cumbersome notations, we shall focus our attention on (2). However, see Section 4.1 for some outcomes in the nonautonomous case.

We point out that an obvious motivation to consider (2) relies on the fact that all results on a priori bounds concerning (2) apply to the general systems (1) provided,

$$H_v(x, u, v) \geq v^p,$$

$$H_u(x, u, v) \geq u^q.$$

It is known that if instead of dealing with system (2), we study the scalar model inequality

$$\mathrm{div}\mathscr{A}(x, u, \nabla u) \geq u^q \quad on \quad \mathbb{R}^N, u \geq 0, \quad on \quad \mathbb{R}^N, \tag{3}$$

then, under suitable assumptions on the differential operator and a natural super-homogeneity hypothesis on q, the problem admits only the trivial weak solution.

These kind of Liouville theorems were first proved in [26] under strong assumptions on the differential operators and on the regularity of the possible weak solutions.

Recently, these results have been extended in different directions to a wide class of differential operators and nonlinearities in Farina and Serrin [15], [16] for σ-regular distribution solutions which are *locally bounded* and may *change sign.*

The *locally boundedness* assumption on the solutions has been recently removed in D'Ambrosio and Mitidieri [12].

The non coercive counterpart of (2), that is

$$\begin{cases} -L(u) \geq v^p \quad on \quad \mathbb{R}^N, \\ \\ -L(v) \geq u^q \quad on \quad \mathbb{R}^N, \\ \\ u \geq 0, \quad v \geq 0 \quad on \quad \mathbb{R}^N, \end{cases} \tag{4}$$

has also been widely studied. Mitidieri in [24] studies the solutions of (4) with $L = \Delta$, proving the existence of the so-called critical Serrin's curve,

$$\max\left\{\frac{p+1}{pq-1}, \frac{q+1}{pq-1}\right\} = \frac{N-2}{2}. \tag{S}$$

If the parameters (p, q) satisfy

$$\max\left\{\frac{p+1}{pq-1}, \frac{q+1}{pq-1}\right\} \geq \frac{N-2}{2},$$

then the solutions are trivial.

Dealing with the system of equations (instead of inequalities), the same kind of result has been proved for radial solutions in [23] by showing the existence of another *sharp* critical Hardy-Littlewood-Sobolev curve,

$$\frac{1}{p+1}+\frac{1}{q+1}=\frac{N-2}{N}. \tag{HLS}$$

Some contributions on the so called (HLS) conjecture, see Caristi, D'Ambrosio and Mitidieri [6], for non radial solutions has been obtained by Serrin and Zou [32]. See also Busca and Manasevich [4], Polacik, Quittner and Souplet [29]. For ground breaking results on this conjecture see Souplet [33]. Related results have been recently obtained by Phan [28].

Later on, problem (4) has been studied in [25] for more general operators. Further recent interesting contributions on (4) and related generalizations can be found in Filippucci [17], [18], and for more general nonlinearities, in D'Ambrosio and Mitidieri [13] and Colasuonno, D'Ambrosio and Mitidieri [8]. Problems of the type (4) has been studied also in [6] for linear operators L of higher order as well as pseudodifferential operators. In [6] the authors use and develop a technique introduced in [14] and based on integral representation of the solutions. Again, they find critical curves for systems of integral equations and inequalities similar to (S) and (HLS). These kind of systems are related to the double weighted Hardy-Littlewood-Sobolev inequality studied by Stein and Weiss [34].

In this paper we use a technique essentially rooted on the classical test functions method and on some ideas introduced in [12].

The main goal of this paper is to study weakly coercive systems of the form (2) in the framework of Carnot groups for σ-regular solutions (see Definition 2.2 below) which are not necessarily *locally bounded*. See the next section for essential preliminaries. We refer the interested reader to [9] and the references therein, for results related to the scalar case on Carnot groups. To the best of our knowledge, studies concerning coercive systems of type (2) in Carnot groups setting are not present in the literature. For the non coercive system see [13].

As a sample of our results in the Euclidean case we have the following.

Theorem 1.1 *Let $\mathscr{A}_i$ be* **W**-*p*-**C** *with $p_i > 1$ and $q_i > p_i - 1$ ($i = 1, 2$). Let (u, v) be a solution of*

$$\begin{cases} \mathrm{div}\mathscr{A}_1(x,u,\nabla u) \geq v^{q_2} \quad on \quad \mathbb{R}^N, \\ \\ \mathrm{div}\mathscr{A}_2(x,v,\nabla v) \geq u^{q_1} \quad on \quad \mathbb{R}^N, \\ \\ u \geq 0, \quad v \geq 0 \quad on \quad \mathbb{R}^N. \end{cases} \tag{5}$$

If

$$\max\left\{q_1\frac{p_2q_2+p_1(p_2-1)}{q_1q_2-(p_1-1)(p_2-1)}+p_1,\ q_2\frac{p_1q_1+p_2(p_1-1)}{q_1q_2-(p_1-1)(p_2-1)}+p_2\right\}\geq N, \tag{6}$$

then $u\equiv v\equiv 0$[1].

As a special case of the above theorem we have,

Corollary 1.2 *Let* $\mathscr{A}_i$ *be* $\mathbf{W}$-p-$\mathbf{C}$ *with* $p_i>1$ *and* $q_i>p_i-1$ $(i=1,2)$*. Let* (u,v) *be a weak solution of (5).*

1. *If* $p_1=p_2=p$*, then* $u\equiv v\equiv 0$ *provided*

$$\max\left\{q_1\frac{q_2+p-1}{q_1q_2-(p-1)^2}, q_2\frac{q_1+p-1}{q_1q_2-(p-1)^2}\right\}\geq\frac{N-p}{p}.$$

   In particular if $N\leq 2p$ *then* $u\equiv v\equiv 0$ *for any* $q_1,q_2>p-1$.

2. *If* $p_1=p_2=p$ *and* $q_1=q_2=q$*, then* $u\equiv v\equiv 0$ *provided* $q(N-2p)\leq(N-p)(p-1)$.

3. *If* $p_1=p_2=2$ *and* $N\geq 5$*, then* $u\equiv v\equiv 0$ *provided* $\max\{q_1,q_2\}\geq q_1q_2\frac{N-4}{2}-\frac{N-2}{2}$.

4. *If* $p_1=p_2=2$ *and* $N\leq 4$*, then* $u\equiv v\equiv 0$.

   In particular if $L_1=L_2$ *is the mean curvature operator and* $N\leq 4$*, then* $u\equiv v\equiv 0$.

By requiring more regularity on one of the operators L_1 or L_2, we can deduce Liouville theorems under the only assumption that the exponents satisfy $q_i>p_i-1$.

Theorem 1.3 *Let* $\mathscr{A}_i$ *be* $\mathbf{W}$-p-$\mathbf{C}$ *with* $p_i>1$ *and* $q_i>p_i-1$ $(i=1,2)$*. Let* (u,v) *be a weak solution of (5).*

1. *Assume that* $\mathscr{A}_1$ *(or* $\mathscr{A}_2$*) is* $\mathbf{S}$-p-$\mathbf{C}$*. Then* $u\equiv v\equiv 0$.

2. *Assume that* u *and* $\mathscr{A}_1$ *are radial (or* v *and* $\mathscr{A}_2$*). Then* $u\equiv v\equiv 0$.

It is worth noting that in this paper we shall perform our analysis for nonnegative solutions of systems of the form (5). This is due to the fact that, if we consider a slightly more general system involving nonlinearities that change sign, one cannot expect to reduce the original problem via Kato's inequality (see [12]) for details) to the study of solutions with non negative components as the following simple result shows.

[1]Throughout this paper we will write $u\equiv 0$ for $u=0$ a.e. on $\mathbb{R}^N$.

Theorem 1.4 *Let (u, v) be a weak solution of*

$$\begin{cases} \Delta_p u = |v|^{q-1} v & on \quad \mathbb{R}^N, \\ \\ \Delta_p v = |u|^{q-1} u & on \quad \mathbb{R}^N, \end{cases} \tag{7}$$

with $q > 1$ and $2 \geq p > 1$. Then $u = -v$ and u and v are solution of the scalar equation

$$-\Delta_p u = |u|^{q-1} u \quad on \quad \mathbb{R}^N.$$

In particular (7) does not admit nonnegative nontrivial solutions.

From the above result one can infer an analogous result for system like (7) where the operator Δ_p is replaced by $-\Delta_p$. See Corollary 5.6 below.

Theorem 1.5 *Let (u, v) be a weak solution of*

$$\begin{cases} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = |v|^{q-1} v & on \quad \mathbb{R}^N, \\ \\ \operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) = |u|^{q-1} u & on \quad \mathbb{R}^N, \end{cases} \tag{8}$$

with $q > 1$. Then $u = -v$ and u and v are solution of the scalar equation

$$-\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = |u|^{q-1} u \quad on \quad \mathbb{R}^N.$$

In particular (8) does not admit nonnegative nontrivial solutions.

This paper is organized as follows. In the next section we introduce some notations and preliminaries. In Section 3 we prove the main estimates on the solutions of (5). The related Liouville theorems are studied in Section 4. Some results for nonautonomous systems are in Section 4.1. Section 5 deals with the study of changing sign solutions of some special case of coercive systems. We end the paper with some open questions.

2 Notations and definitions

In this paper ∇ and $|\cdot|$ stand respectively for the usual gradient in $\mathbb{R}^N$ and the Euclidean norm. $\Omega \subset \mathbb{R}^N$ denotes an open set.

In this paper ∇ and $|\cdot|$ stand respectively for the usual gradient in $\mathbb{R}^N$ and the Euclidean norm.

Let $\mu \in \mathscr{C}(\mathbb{R}^N;\mathbb{R}^l)$ be a matrix $\mu := (\mu_{ij})$, $i = 1,\ldots,l$, $j = 1,\ldots,N$. For $i = 1,\ldots,l$, let X_i and its formal adjoint X_i^* be defined as

$$X_i := \sum_{j=1}^{N} \mu_{ij}(\xi)\frac{\partial}{\partial \xi_j}, \qquad X_i^* := -\sum_{j=1}^{N} \frac{\partial}{\partial \xi_j}\left(\mu_{ij}(\xi)\cdot\right), \tag{9}$$

and let ∇_L be the vector field defined by $\nabla_L := (X_1,\ldots,X_l)^T = \mu\nabla$ and $\nabla_L^* := (X_1^*,\ldots,X_l^*)^T$.

For any vector field $h = (h_1,\ldots,h_l)^T \in \mathscr{C}^1(\Omega,\mathbb{R}^l)$, we shall use the following notation $\mathrm{div}_L(h) := \mathrm{div}(\mu^T h)$, that is

$$\mathrm{div}_L(h) = -\sum_{i=1}^{l} X_i^* h_i = -\nabla_L^* \cdot h. \tag{10}$$

An assumption that we shall made (which actually is an assumption on the matrix μ) is that the operator

$$\Delta_G u = \mathrm{div}_L(\nabla_L u)$$

is a canonical sub-Laplacian on a Carnot group (see below for a more precise meaning). The reader, which is not acquainted with these structures, can think to the special case of $\mu = I$, the identity matrix in $\mathbb{R}^N$, that is the usual Laplace operator in Euclidean setting. Our results are new even in the Euclidean case.

We quote some facts on Carnot groups and refer the interested reader to the books of Bonfiglioli, Lanconelli and Uguzzoni [3], Folland and Stein [20] and to Folland [19] for more detailed information on this subject.

A Carnot group is a connected, simply connected, nilpotent Lie group $\mathbb{G}$ of dimension N with graded Lie algebra $\mathcal{G} = V_1 \oplus \cdots \oplus V_r$ such that $[V_1, V_i] = V_{i+1}$ for $i = 1 \ldots r-1$ and $[V_1, V_r] = 0$. Such an integer r is called the *step* of the group. We set $l = n_1 = \dim V_1$, $n_2 = \dim V_2, \ldots, n_r = \dim V_r$. A Carnot group $\mathbb{G}$ of dimension N can be identified, up to an isomorphism, with the structure of a *homogeneous Carnot Group* $(\mathbb{R}^N, \circ, \delta_R)$ defined as follows; we identify $\mathbb{G}$ with $\mathbb{R}^N$ endowed with a Lie group law $\circ$. We consider $\mathbb{R}^N$ split in r subspaces $\mathbb{R}^N = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_r}$ with $n_1 + n_2 + \cdots + n_r = N$ and $\xi = (\xi^{(1)},\ldots,\xi^{(r)})$ with $\xi^{(i)} \in \mathbb{R}^{n_i}$. We shall assume that for any $R > 0$ the dilation $\delta_R(\xi) = (R\xi^{(1)}, R^2\xi^{(2)},\ldots,R^r\xi^{(r)})$ is a Lie group automorphism. The Lie algebra of left-invariant vector fields on $(\mathbb{R}^N,\circ)$ is $\mathcal{G}$. For $i = 1,\ldots,n_1 = l$ let X_i be the unique vector

field in $\mathcal{G}$ that coincides with $\partial/\partial\xi_i^{(1)}$ at the origin. We require that the Lie algebra generated by $X_1,\ldots,X_l$ is the whole $\mathcal{G}$.

We denote with ∇_L the vector field $\nabla_L := (X_1,\ldots,X_l)^T$ and we call it *horizontal vector field* and by div_L the formal adjoint on ∇_L, that is (10). Moreover, the vector fields $X_1,\ldots,X_l$ are homogeneous of degree 1 with respect to δ_R and in this case $Q = \sum_{i=1}^r i\, n_i = \sum_{i=1}^r i\, \mathrm{dim} V_i$ is called the *homogeneous dimension* of $\mathbb{G}$. The *canonical sub-Laplacian* on $\mathbb{G}$ is the second order differential operator defined by

$$\Delta_G = \sum_{i=1}^l X_i^2 = \mathrm{div}_L(\nabla_L\cdot)$$

and for $p > 1$ the p-sub-Laplacian operator is

$$\Delta_{G,p} u := \sum_{i=1}^l X_i(|\nabla_L u|^{p-2} X_i u) = \mathrm{div}_L(|\nabla_L u|^{p-2} \nabla_L u).$$

Since $X_1,\ldots,X_l$ generate the whole $\mathcal{G}$, the sub-Laplacian Δ_G satisfies the Hörmander hypoellipticity condition.

A nonnegative continuous function $\nu : \mathbb{R}^N \to \mathbb{R}_+$ is called a *homogeneous norm* on $\mathbb{G}$, if $\nu(\xi^{-1}) = \nu(\xi)$, $\nu(\xi) = 0$ if and only if $\xi = 0$, and it is homogeneous of degree 1 with respect to δ_R (i.e. $\nu(\delta_R(\xi)) = R\nu(\xi)$). A homogeneous norm ν defines on $\mathbb{G}$ a *pseudo-distance* defined as $d(\xi,\eta) := \nu(\xi^{-1}\eta)$, which in general is not a distance. If ν and $\tilde{\nu}$ are two homogeneous norms, then they are equivalent, that is, there exists a constant $C > 0$ such that $C^{-1}\nu(\xi) \le \tilde{\nu}(\xi) \le C\nu(\xi)$. Let ν be a homogeneous norm, then there exists a constant $C > 0$ such that $C^{-1}|\xi| \le \nu(\xi) \le C|\xi|^{1/r}$, for $\nu(\xi) \le 1$. An example of homogeneous norm is $\nu(\xi) := \left(\sum_{i=1}^r |\xi_i|^{2r!/i}\right)^{1/2r!}$.

Notice that if ν is a homogeneous norm differentiable a.e., then $|\nabla_L \nu|$ is homogeneous of degree 0 with respect to δ_R; hence $|\nabla_L \nu|$ is bounded.

We notice that in a Carnot group, the Haar measure coincides with the Lebesgue measure.

Special examples of Carnot groups are the Euclidean spaces $\mathbb{R}^Q$. Moreover, if $Q \le 3$ then any Carnot group is the ordinary Euclidean space $\mathbb{R}^Q$.

The simplest nontrivial example of a Carnot group is the Heisenberg group $\mathbb{H}^1 = \mathbb{R}^3$. For an integer $n \ge 1$, the Heisenberg group $\mathbb{H}^n$ is defined as follows: let $\xi = (\xi^{(1)}, \xi^{(2)})$ with $\xi^{(1)} := (x_1,\ldots,x_n,y_1,\ldots,y_n)$ and $\xi^{(2)} := t$. We endow $\mathbb{R}^{2n+1}$ with the group law $\hat{\xi} \circ \tilde{\xi} := (\hat{x} + \tilde{x}, \hat{y} + \tilde{y}, \hat{t} + \tilde{t} + 2\sum_{i=1}^n (\tilde{x}_i \hat{y}_i - \hat{x}_i \tilde{y}_i))$. We consider the vector fields

$$X_i := \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \; Y_i := \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, \qquad \text{for } i = 1,\ldots,n,$$

and the associated Heisenberg gradient $\nabla_H := (X_1,\ldots,X_n,Y_1,\ldots,Y_n)^T$. The Kohn Laplacian Δ_H is then the operator defined by $\Delta_H := \sum_{i=1}^n X_i^2 + Y_i^2$. The family of dilations

is given by $\delta_R(\xi) := (Rx, Ry, R^2 t)$ with homogeneous dimension $Q = 2n+2$. In $\mathbb{H}^n$ a canonical homogeneous norm is defined as $|\xi|_H := \left(\left(\sum_{i=1}^n x_i^2 + y_i^2\right)^2 + t^2 \right)^{1/4}$.

In what follows we shall fix a homogeneous norm ν and for $R > 0$, B_R stands for the ball of radius $R > 0$ generated by ν, that is $B_R := \{x : \nu(x) < R\}$ and A_R is the annulus $B_{2R} \setminus \overline{B_R}$. Therefore, by using the dilation δ_R and the fact that the Jacobian of δ_R is R^Q, we have

$$|B_R| = \int_{B_R} dx = R^Q \int_{B_1} dx = w_\nu R^Q \quad \text{and} \quad |A_R| = w_\nu (2^Q - 1) R^Q,$$

where w_ν is the Lebesgue measure of the unit ball B_1 in $\mathbb{R}^N$.

Throughout the paper $p \geq 1$ and we denote by $W^{1,p}_{L,loc}$ the space

$$W^{1,p}_{L,loc}(\Omega) := \{ u \in L^p_{loc}(\Omega) \ : \ |\nabla_L u| \in L^p_{loc}(\Omega) \} .$$

Clearly, if $\nabla_L = \nabla$, that is the Carnot group is the trivial Euclidean space $\mathbb{R}^N$, then $W^{1,p}_{L,loc}(\Omega)$ coincides with the usual Sobolev space $W^{1,p}_{loc}(\Omega)$.

In what follows we shall assume that $\mathscr{A} \colon \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^l$ is a Caratheodory function, that is for each $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^l$ the function $\mathscr{A}(\cdot, t, \xi)$ is measurable; and for a.e. $x \in \mathbb{R}^N$, $\mathscr{A}(x, \cdot, \cdot)$ is continuous.

We consider operators L "generated" by $\mathscr{A}$, that is

$$L(u)(x) = \operatorname{div}_L \left(\mathscr{A}(x, u(x), \nabla_L u(x)) \right) .$$

Our model cases are the p-Laplacian operator, the mean curvature operator and some related generalizations. See Examples 2.4 below.

Definition 2.1 *Let $\mathscr{A} \colon \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^l$ be a Caratheodory function. The function $\mathscr{A}$ is called* weakly elliptic *if it generates a weakly elliptic operator L i.e.*

$$\begin{gathered} \mathscr{A}(x,t,\xi) \cdot \xi \geq 0 \ \ \textit{for each } x \in \mathbb{R}^N,\ t \in \mathbb{R},\ \xi \in \mathbb{R}^l, \\ \mathscr{A}(x,0,\xi) = 0 \ \ \textit{or} \ \ \mathscr{A}(x,t,0) = 0 \end{gathered} \tag{WE}$$

Let $p \geq 1$, the function $\mathscr{A}$ is called **W**-p-**C** *(weakly-p-coercive), if $\mathscr{A}$ is (WE) and it generates a weakly-p-coercive operator L, i.e. if there exists a constant $k_2 > 0$ such that*

$$(\mathscr{A}(x,t,\xi) \cdot \xi)^{p-1} \geq k_2^{p-1} \left| \mathscr{A}(x,t,\xi) \right|^p \ \text{ for each } x \in \mathbb{R}^N,\ t \in \mathbb{R},\ \xi \in \mathbb{R}^l. \tag{$\mathbf{W}$-p-$\mathbf{C}$}$$

Let $p > 1$, the function $\mathscr{A}$ is called **S**-p-**C** *(strongly-p-coercive), if there exist $k_1, k_2 > 0$ constants such that*

$$(\mathscr{A}(x,t,\xi) \cdot \xi) \geq k_1 \left| \xi \right|^p \geq k_2 \left| \mathscr{A}(x,t,\xi) \right|^{p'} \ \text{ for each } x \in \mathbb{R}^N,\ t \in \mathbb{R},\ \xi \in \mathbb{R}^l. \tag{$\mathbf{S}$-p-$\mathbf{C}$}$$

For additional information on (WE), **W**-p-**C** and **S**-p-**C** operators see [31, 2, 27, 12] and the references therein.

From now on, if not otherwise specified, $\mathscr{A}$, $\mathscr{A}_1$ and $\mathscr{A}_2$ stands for **W**-p-**C** function with indices p, p_1 and p_2 respectively.

Definition 2.2 *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $f : \Omega \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}$ be a Caratheodory function. Let $p \geq 1$. We say that $u \in W^{1,p}_{L,loc}(\Omega)$ is a* weak solution *of*

$$\mathrm{div}_L \left(\mathscr{A}(x, u, \nabla_L u)\right) \geq f(x, u, \nabla_L u) \qquad \text{on} \ \ \Omega,$$

if $\mathscr{A}(\cdot, u, \nabla_L u) \in L^{p'}_{loc}(\Omega)$, $f(\cdot, u, \nabla_L u) \in L^1_{loc}(\Omega)$, and for any nonnegative $\phi \in \mathscr{C}^1_0(\Omega)$ we have

$$-\int_\Omega \mathscr{A}(x, u, \nabla_L u) \cdot \nabla_L \phi \geq \int_\Omega f(x, u, \nabla_L u)\phi.$$

In the literature this kind of solutions are called σ-regular solutions, see for instance [30].

Definition 2.3 *Let $p_1, p_2 \geq 1$ and let $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ be Caratheodory functions. We say that $(u, v) \in W^{1,p_1}_{L,loc}(\Omega) \times W^{1,p_2}_{L,loc}(\Omega)$ is a* weak solution *of*

$$\begin{cases} \mathrm{div}_L(\mathscr{A}_1(x, u, \nabla_L u)) \geq f(x, v) & \text{on } \Omega, \\ \\ \mathrm{div}_L(\mathscr{A}_2(x, v, \nabla_L v)) \geq g(x, u) & \text{on } \Omega. \end{cases} \tag{11}$$

if $\mathscr{A}_i(\cdot, u, \nabla_L u) \in L^{p'_i}_{loc}(\Omega)$ $(i = 1, 2)$, $f(x, v, \nabla_L v, u, \nabla_L u), g(x, v, \nabla_L v, u, \nabla_L u) \in L^1_{loc}(\Omega)$, and for any nonnegative $\phi \in \mathscr{C}^1_0(\Omega)$ we have

$$\begin{cases} -\displaystyle\int_\Omega \mathscr{A}_1(x, u, \nabla_L u) \cdot \nabla_L \phi \geq \int_\Omega f(x, v)\phi, \\ \\ -\displaystyle\int_\Omega \mathscr{A}_2(x, v, \nabla_L v) \cdot \nabla_L \phi \geq \int_\Omega g(x, u)\phi. \end{cases} \tag{12}$$

Example 2.4

1. *Let $p > 1$. The sub-p-Laplacian operator defined on suitable functions u by,*

   $$\Delta_{G,p} u = \mathrm{div}_L \left(|\nabla_L u|^{p-2} \nabla_L u\right)$$

   is an operator generated by $\mathscr{A}(x, t, \xi) := |\xi|^{p-2}\xi$ which is **S**-p-**C**.

2. *If $\mathscr{A}$ is of mean curvature type, that is $\mathscr{A}$ can be written as $\mathscr{A}(x, t, \xi) := A(|\xi|)\xi$ with $A : \mathbb{R} \to \mathbb{R}$ a positive bounded continuous function (see [26, 2]), then $\mathscr{A}$ is* **W-2-C**.

3. *The mean curvature operator in non parametric form*

$$Tu := \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right),$$

is generated by $\mathscr{A}(x,t,\xi) := \frac{\xi}{\sqrt{1+|\xi|^2}}$. *In this case* $\mathscr{A}$ *is* $\mathbf{W}$-p-$\mathbf{C}$ *with* $1 \le p \le 2$ *and of mean curvature type but it is not* $\mathbf{S}$-2-$\mathbf{C}$.

4. *Let* $m > 1$. *The operator*

$$T_m u := \operatorname{div}\left(\frac{|\nabla u|^{m-2}\nabla u}{\sqrt{1+|\nabla u|^m}}\right)$$

is $\mathbf{W}$-p-$\mathbf{C}$ *for* $m \ge p \ge m/2$.

5. *Let* $p > 1$ *and define*

$$Lu := \sum_{i=1}^{N} \partial_i\left(|\partial_i u|^{p-2}\partial_i u\right).$$

The operator L *is* $\mathbf{S}$-p-$\mathbf{C}$.

Observe that if $\mathscr{A}$ is **S-**p**-C** and ∇_L is the Euclidean gradient ∇, then the following important Serrin-Trudinger inequality holds [31, 36, 37]. For a proof in the Euclidean setting see, for instance, Maly and Ziemer [22], in the case of Carnot groups see Capogna, Danielli and Garofalo [5].

Lemma 2.5 *(Weak Harnack Inequality) Let* $\mathscr{A}$ *be* $\mathbf{S}$-p-$\mathbf{C}$. *If* $u \in W^{1,p}_{L,loc}(\mathbb{R}^N)$ *is a nonnegative weak solution of*

$$\operatorname{div}_L(\mathscr{A}(x,u,\nabla_L u)) \ge 0, \tag{13}$$

then for any $\sigma > 0$ *there exists* $c_H > 0$ *independent of* u *such that for any* $R > 0$ *we have*

$$\left(\frac{1}{|B_R|}\int_{B_R} u^\sigma dx\right)^{\frac{1}{\sigma}} \ge\ c_H \operatorname{esssup}_{B_{\frac{R}{2}}} u. \tag{14}$$

Motivated by the above result, we introduce the following.

Definition 2.6 (WH) *We say that the* weak Harnack inequality with exponent $\sigma > 0$ *holds for* $\mathscr{A}$ *if for any weak solution* u *of*

$$\operatorname{div}_L(\mathscr{A}(x,u,\nabla_L u)) \ge 0, \quad u \ge 0, \quad \text{on } \Omega \tag{15}$$

and any $R > 0$ *such that* $B_{2R} \subset \Omega$ *we have*

$$\left(\frac{1}{|B_R|}\int_{B_R} u^\sigma dx\right)^{\frac{1}{\sigma}} \ge\ c_H \operatorname{esssup}_{B_{\frac{R}{2}}} u, \tag{WH}$$

with $c_H > 0$ *independent of* u *and* R.

3 A priori universal estimates

In this Section, if not otherwise specified, $\mathscr{A}, \mathscr{A}_1$ and $\mathscr{A}_2$ are **W**-p-**C** with indices $p > 1$, $p_1 > 1$ and $p_2 > 1$ respectively.

The following lemma is a slight variation of a result proved in [12]. For easy reference and for reader's convenience we shall include the detailed proof.

Lemma 3.1 *Let $g \in L^1_{loc}(\Omega)$ be nonnegative and let $u \in W^{1,p}_{L,loc}(\Omega)$ be a weak solution of*

$$\mathrm{div}_L\left(\mathscr{A}(x,u,\nabla_L u)\right) \geq g, \qquad u \geq 0, \quad on \;\; \Omega. \tag{16}$$

Let $s > 0$. If $u^{s+p-1} \in L^1_{loc}(\Omega)$, then

$$gu^s, \quad \mathscr{A}(x,u,\nabla_L u)\cdot\nabla_L u \; u^{s-1} \in L^1_{loc}(\Omega) \tag{17}$$

and for any nonnegative $\phi \in \mathscr{C}^1_0(\Omega)$ we have,

$$\int_\Omega gu^s\phi + c_1 s\int_\Omega \mathscr{A}(x,u,\nabla_L u)\cdot\nabla_L u\; u^{s-1}\phi \leq c_2 s^{1-p}\int_\Omega u^{s+p-1}\frac{|\nabla_L\phi|^p}{\phi^{p-1}}, \tag{18}$$

where $c_1 = 1 - \frac{\epsilon^{p'}}{p'k_2} > 0$, $c_2 = \frac{p^p}{p\epsilon^p}$ and $\epsilon > 0$ is sufficiently small.

Remark 3.2 *i) Notice that from the above result it follows that if $u \in W^{1,p}_{L,loc}(\Omega)$ is a weak solution of (16), then $g\,u \in L^1_{loc}(\Omega)$.*

ii) The above lemma still holds if we replace the function $g \in L^1_{loc}(\Omega)$ with a regular Borel measure on Ω.

Remark 3.3 *From Lemma 3.1 we deduce that if $p > 1$ and u is a weak solution of (16) and u vanishes on a set $O \subset \Omega$ of positive measure, then $\mathscr{A}(x,u,\nabla_L u)$ must vanishes, modulo sets of measure zero, on O. Indeed, we can choose $0 < s < 1$ so that u^{s-1} is infinity on O. Since (17) holds then necessarily $\mathscr{A}(x,u,\nabla_L u)\cdot\nabla_L u = 0$ on O. Now using the fact that $\mathscr{A}$ is* **W**-p-**C** *with $p > 1$, we obtain the claim.*

Proof. Let $\gamma \in \mathscr{C}^1(\mathbb{R})$ be a bounded nonnegative function with bounded nonnegative first derivative and let $\phi \in \mathscr{C}^1_0(\Omega)$ be a nonnegative test function.

Applying Lemma 2.2, of [12] it follows that,

$$\begin{aligned}
\int_\Omega g\gamma(u)\phi + \int_\Omega \mathscr{A}(x,u,\nabla_L u)\cdot\; \nabla_L u\;\gamma'(u)\phi &\leq -\int_\Omega \mathscr{A}(x,u,\nabla_L u)\cdot\nabla_L\phi\;\gamma(u)\\
\leq \int_\Omega |\mathscr{A}(x,u,\nabla_L u)|\;\; |\nabla_L\phi|\,\gamma(u)&\\
\leq \left(\int_\Omega |\mathscr{A}(x,u,\nabla_L u)|^{p'}\gamma'(u)\phi\right)^{1/p'}\left(\int_\Omega \frac{\gamma(u)^p}{\gamma'(u)^{p-1}}\frac{|\nabla_L\phi|^p}{\phi^{p-1}}\right)^{1/p}&\\
\leq \frac{\epsilon^{p'}}{p'k_2}\int_\Omega \mathscr{A}(x,u,\nabla_L u)\cdot\nabla_L u\;\gamma'(u)\phi + \frac{1}{p\epsilon^p}\int_\Omega \frac{\gamma(u)^p}{\gamma'(u)^{p-1}}\frac{|\nabla_L\phi|^p}{\phi^{p-1}},&
\end{aligned}$$

where $\epsilon > 0$. Notice that all integrals are well defined provided $\frac{\gamma(u)^p}{\gamma'(u)^{p-1}} \in L^1_{loc}(\Omega)$. With a suitable choice of $\epsilon > 0$, for any nonnegative $\phi \in \mathscr{C}^1_0(\Omega)$ and $\gamma \in \mathscr{C}^1(\mathbb{R})$ such that $\frac{\gamma(u)^p}{\gamma'(u)^{p-1}} \in L^1_{loc}(\Omega)$, we have,

$$\int_\Omega g\gamma(u)\phi + c_1 \int_\Omega \mathscr{A}\, \nabla_L u \gamma'(u)\phi \leq \frac{1}{p\epsilon^p} \int_\Omega \frac{\gamma(u)^p}{\gamma'(u)^{p-1}} \frac{|\nabla_L \phi|^p}{\phi^{p-1}}. \tag{19}$$

Now for $s > 0$, $1 > \delta > 0$ and $n \geq 1$, define

$$\gamma_n(t) := \begin{cases} (t+\delta)^s & if\ 0 \leq t < n-\delta, \\ \\ cn^s - \dfrac{s}{\beta-1} n^{\beta+s-1}(t+\delta)^{1-\beta} & if\ t \geq n-\delta, \end{cases} \tag{20}$$

where $c := \frac{\beta-1+s}{\beta-1}$ and $\beta > 1$ will be chosen later. Clearly $\gamma_n \in \mathscr{C}^1$,

$$\gamma_n'(t) = \begin{cases} s(t+\delta)^{s-1} & if\ 0 \leq t < n-\delta, \\ \\ sn^{\beta+s-1}(t+\delta)^{-\beta} & if\ t \geq n-\delta, \end{cases}$$

and γ_n, γ_n' are nonnegative and bounded with $||\gamma_n||_\infty = cn^s$ and $||\gamma_n'||_\infty = sn^{s-1}$. Moreover

$$\frac{\gamma_n(t)^p}{\gamma_n'(t)^{p-1}} = \begin{cases} s^{1-p}(t+\delta)^{s+p-1} & for\ t < n-\delta, \\ \\ \theta(t,n) & for\ t \geq n-\delta, \end{cases}$$

where

$$\theta(t,n) := \frac{(cn^s - \frac{s}{\beta-1} n^{\beta+s-1}(t+\delta)^{1-\beta})^p}{(sn^{\beta+s-1}(t+\delta)^{-\beta})^{p-1}} \leq (cn^s)^p s^{1-p} n^{-(\beta+s-1)(p-1)}\, (t+\delta)^{\beta(p-1)}.$$

Choosing $\beta := \frac{s+p-1}{p-1}$ we have $c = p$, and

$$\theta(t,n) \leq p^p s^{1-p} n^{sp-(\beta+s-1)(p-1)}(t+\delta)^{s+p-1} = p^p s^{1-p}(t+\delta)^{s+p-1}.$$

Therefore, for $t \geq 0$ we have,

$$\frac{\gamma_n(t)^p}{\gamma_n'(t)^{p-1}} \leq p^p s^{1-p}(t+\delta)^{s+p-1}.$$

Since by assumption $u^{s+p-1} \in L^1_{loc}(\Omega)$, from (19) with $\gamma = \gamma_n$, it follows that

$$\int_\Omega g\gamma_n(u)\phi + c_1 \int_\Omega \mathscr{A}(x,u,\nabla_L u) \cdot \nabla_L u\ \gamma_n'(u)\phi \leq \frac{p^p s^{1-p}}{p\epsilon^p} \int_\Omega (u+\delta)^{s+p-1} \frac{|\nabla_L \phi|^p}{\phi^{p-1}}.$$

Noticing that $\gamma_n(t) \to (t+\delta)^s$ and $\gamma_n'(t) \to s(t+\delta)^{s-1}$ as $n \to +\infty$, $g \geq 0$ and $\mathscr{A} \cdot \nabla_L u \geq 0$, by Beppo Levi theorem we obtain

$$\int_\Omega g\,(u+\delta)^s\phi + c_1 s \int_\Omega \mathscr{A}(x,u,\nabla_L u) \cdot \nabla_L u\;(u+\delta)^{s-1}\phi \leq c_2 s^{1-p} \int_\Omega (u+\delta)^{s+p-1}\frac{|\nabla_L\phi|^p}{\phi^{p-1}},$$

which, by letting $\delta \to 0$, completes the proof. $\square$

Remark 3.4 *The assumption $u^{s+p-1} \in L^1_{loc}(\Omega)$, is not needed for the validity of the statement (17). Indeed what really matters is the assumption $u^{s+p-1} \in L^1_{loc}(S)$. Here S is the support of $\nabla_L\phi$. This observation will be useful when dealing with inequalities on unbounded set.*

Lemma 3.5 *Let $g \in L^1_{loc}(\Omega)$ be nonnegative and let $u \in W^{1,p}_{L,loc}(\Omega)$ be a weak solution of (16). Let $q > p-1$. If $u^q \in L^1_{loc}(\Omega)$, then*

$$gu^{q-p+1}, \quad \mathscr{A}(x,u,\nabla_L u) \cdot \nabla_L u\; u^{q-p} \in L^1_{loc}(\Omega) \tag{21}$$

and for any $\varphi \in \mathscr{C}^1_0(\Omega)$ such that $0 \leq \varphi \leq 1$, we have,

$$\int_\Omega g\varphi^\sigma \leq c_3 \left(\frac{1}{|S|}\int_S u^q\varphi^\sigma\right)^{\frac{p-1}{q}} \left(\frac{1}{|S|}\int_S |\nabla_L\varphi|^\sigma\right)^{\frac{p}{\sigma}} |S|, \tag{22}$$

where S is the support of $\nabla_L\varphi$, $c_3 := \frac{\sigma^p}{s^{p-1}}\left(\frac{c_2}{c_1 k_2}\right)^{1/p'}$ with $\sigma \geq \frac{pq}{q-p+1-s}$, $0 < s < \min\{1, q-p+1\}$ and c_1, c_2 as in the above Lemma 3.1.

Proof. Let $s > 0$ be such that $q \geq s+p-1$. From Lemma 3.1, for any nonnegative $\phi \in \mathscr{C}^1_0(\omega)$ we have

$$\int_\Omega gu^s\phi + c_1 s \int_\Omega \mathscr{A}(x,u,\nabla_L u) \cdot \nabla_L u\; u^{s-1}\phi \leq c_2 s^{1-p} \int_\Omega u^{s+p-1}\frac{|\nabla_L\phi|^p}{\phi^{p-1}}. \tag{23}$$

Now, let $0 < s < \min\{1, q-p+1\}$. By definition of weak solution and Hölder's inequality with exponent p', taking into account that $\mathscr{A}$ is **W**-p-**C** and from (23) we get,

$$\begin{aligned}
\int_\Omega g\phi &\leq \int_S |\mathscr{A}(x,u,\nabla_L u)|\,|\nabla_L\phi| = \int_S |\mathscr{A}|\, u^{\frac{s-1}{p'}}\phi^{\frac{1}{p'}}\;\; |\nabla_L\phi|\, u^{\frac{1-s}{p'}}\phi^{-\frac{1}{p'}} && (24)\\
&\leq \frac{1}{k_2^{1/p'}}\left(\int_S \mathscr{A}(x,u,\nabla_L u)\cdot\nabla_L u\, u^{s-1}\phi\right)^{1/p'}\left(\int_S u^{(1-s)(p-1)}\frac{|\nabla_L\phi|^p}{\phi^{p-1}}\right)^{1/p} && (25)\\
&\leq \frac{1}{k_2^{1/p'}}\left(\frac{c_2}{c_1 s^p}\right)^{1/p'}\left(\int_S u^{s+p-1}\frac{|\nabla_L\phi|^p}{\phi^{p-1}}\right)^{1/p'}\left(\int_S u^{(1-s)(p-1)}\frac{|\nabla_L\phi|^p}{\phi^{p-1}}\right)^{1/p}. && (26)
\end{aligned}$$

Since $q > s+p-1$ and $q > p-1$, applying Hölder inequality to (26) with exponents $\chi := \frac{q}{s+p-1}$ and $y := \frac{q}{(1-s)(p-1)}$, we obtain

$$\int_\Omega g\phi \le c_3' \left(\int_S u^q \phi\right)^\delta \left(\int_S \frac{|\nabla_L \phi|^{p\chi'}}{\phi^{p\chi'-1}}\right)^{\frac{1}{p'\chi'}} \left(\int_S \frac{|\nabla_L \phi|^{py'}}{\phi^{py'-1}}\right)^{\frac{1}{py'}}, \tag{27}$$

where

$$\delta := \frac{1}{\chi p'} + \frac{1}{yp} = \frac{p-1}{q}, \qquad c_3' := \left(\frac{c_2}{k_2 c_1 s^p}\right)^{1/p'}.$$

Next for $\sigma \ge p\chi'$ (notice that, since $p\chi' > py'$ we have $\sigma > py'$) we choose $\phi := \varphi^\sigma$ with $\varphi \in \mathscr{C}_0^1(\Omega)$ such that $0 \le \varphi \le 1$. Setting $S := support(\varphi)$, from (27) it follows that

$$\int_\Omega g\varphi^\sigma \le c_3'\sigma^p \left(\int_S u^q \varphi^\sigma\right)^\delta \left(\frac{1}{|S|}\int_S |\nabla_L \varphi|^\sigma\right)^{\frac{p}{\sigma}} |S|^{1-\delta}, \tag{28}$$

completing the proof of (22). □

Lemma 3.6 *Let $q_1 > p_1 - 1$ and $q_2 > p_2 - 1$. For any $\sigma > 0$ large enough, there exists a constant $c = c(\sigma, q_1, q_2, p_1, p_2, \mathscr{A}_1, \mathscr{A}_2) > 0$ such that if (u, v) is weak solution of*

$$\begin{cases} \operatorname{div}_L(\mathscr{A}_1(x,u,\nabla_L u)) \ge v^{q_2} & \text{on } \Omega, \\ \operatorname{div}_L(\mathscr{A}_2(x,v,\nabla_L v)) \ge u^{q_1} & \text{on } \Omega, \\ v \ge 0, \ u \ge 0, \end{cases} \tag{29}$$

then for any nonnegative $\varphi \in \mathscr{C}_0^1(\Omega)$ such that $||\varphi||_\infty \le 1$, we have

$$\int_\Omega u^{q_1}\varphi^\sigma \le c \left(\frac{1}{|S|}\int_S v^{q_2}\varphi^\sigma\right)^{\frac{p_2-1}{q_2}} \left(\frac{1}{|S|}\int_S |\nabla_L \varphi|^\sigma\right)^{\frac{p_2}{\sigma}} |S| \tag{30}$$

$$\int_\Omega v^{q_2}\varphi^\sigma \le c \left(\frac{1}{|S|}\int_S u^{q_1}\varphi^\sigma\right)^{\frac{p_1-1}{q_1}} \left(\frac{1}{|S|}\int_S |\nabla_L \varphi|^\sigma\right)^{\frac{p_1}{\sigma}} |S| \tag{31}$$

$$\int_\Omega u^{q_1}\varphi^\sigma \le c\ |S| \left(\frac{1}{|S|}\int_S |\nabla_L \varphi|^\sigma\right)^{\frac{1}{\sigma}\frac{p_1\delta_2+p_2}{1-\delta_1\delta_2}} \tag{32}$$

$$\int_\Omega v^{q_2}\varphi^\sigma \le c\ |S| \left(\frac{1}{|S|}\int_S |\nabla_L \varphi|^\sigma\right)^{\frac{1}{\sigma}\frac{p_2\delta_1+p_1}{1-\delta_1\delta_2}}, \tag{33}$$

where $S := support(\varphi)$, $\delta_1 := \frac{p_1-1}{q_1}$ and $\delta_2 := \frac{p_2-1}{q_2}$.

Proof. From Lemma 3.5 we immediately obtain (30) and (31).

Since

$$\int_S v^{q_2}\varphi^\sigma \le \int_\Omega v^{q_2}\varphi^\sigma,$$

by inserting (31) in (30) we get (32). In a similar way we can prove (33). $\square$

Lemma 3.7 *Let $q_1 > p_1 - 1$ and $q_2 > p_2 - 1$. There exists $c > 0$ such that if (u, v) is weak solution of (29), $x \in \Omega$ and $B_{2R}(x) \subset\subset \Omega$, then*

$$\left(\fint_{B_R(x)} v^{q_2}\right)^{1/q_2} \le cR^{-\frac{p_1q_1+p_2(p_1-1)}{q_1q_2-(p_1-1)(p_2-1)}}, \tag{34}$$

$$\left(\fint_{B_R(x)} u^{q_1}\right)^{1/q_1} \le cR^{-\frac{p_2q_2+p_1(p_2-1)}{q_1q_2-(p_1-1)(p_2-1)}}, \tag{35}$$

$$\fint_{B_R(x)} v^{q_2} \le c\left(\fint_{B_{2R}(x)} u^{q_1}\right)^{\frac{p_1-1}{q_1}} R^{-p_1}, \qquad \fint_{B_R(x)} u^{q_1} \le c\left(\fint_{B_{2R}(x)} v^{q_2}\right)^{\frac{p_2-1}{q_2}} R^{-p_2}. \tag{36}$$

Proof. Let $\phi_0 \in \mathscr{C}_0^1(\mathbb{R})$ be such that

$$\phi_0(t) = 0 \ for \ |t| \ge 2, \ \phi_0(t) = 1 \ for \ |t| \le 1, \ and \ 0 \le \phi_0 \le 1.$$

For $R > 0$, we define $\phi_R(y) := \phi_0(\frac{\nu(x^{-1}y)}{R})$.

By choosing $\varphi = \phi_R$ in Lemma 3.6, and observing that

$$\left(\frac{1}{|S|}\int_S |\nabla_L\varphi|^\sigma\right)^{\frac{1}{\sigma}} = \left(\frac{1}{|B_{2R}(x)\setminus B_R(x)|}\int_{B_{2R}(x)\setminus B_R(x)} |\nabla_L\phi_R(y)dy|^\sigma\right)^{\frac{1}{\sigma}} = R^{-1}c(\phi_0, Q, \nu, \sigma),$$

the claim follows from (30), (31), (32) and (33). $\square$

An immediate consequence of the above results are the following universal estimates on the solutions of (29).

Lemma 3.8 *Let $q_1 > p_1 - 1$ and $q_2 > p_2 - 1$. There exists $c > 0$ such that for any (u, v) weak solution of (29), $x \in \Omega$ and $R = dist(x, \partial\Omega)/2$, we have*

$$\left(\fint_{B_R(x)} v^{q_2}\right)^{1/q_2} \le c\ dist(x, \partial\Omega)^{-\frac{p_1q_1+p_2(p_1-1)}{q_1q_2-(p_1-1)(p_2-1)}}, \tag{37}$$

$$\left(\fint_{B_R(x)} u^{q_1}\right)^{1/q_1} \le c\ dist(x, \partial\Omega)^{-\frac{p_2q_2+p_1(p_2-1)}{q_1q_2-(p_1-1)(p_2-1)}},$$

$$\fint_{B_R(x)} v^{q_2} \leq c \left(\fint_{B_{2R}(x)\setminus B_R(x)} u^{q_1} \right)^{\frac{p_1-1}{q_1}} dist(x, \partial\Omega)^{-p_1}, \tag{38}$$

$$\fint_{B_R(x)} u^{q_1} \leq c \left(\fint_{B_{2R}(x)\setminus B_R(x)} v^{q_2} \right)^{\frac{p_2-1}{q_2}} dist(x, \partial\Omega)^{-p_2}. \tag{39}$$

Moreover if (WH) holds for $\mathscr{A}_1$ *(resp.* $\mathscr{A}_2$*), then for a.e.* $x \in \Omega$ *we have*

$$u(x) \leq c\ dist(x, \partial\Omega)^{-\frac{p_2 q_2 + p_1(p_2-1)}{q_1 q_2 - (p_1-1)(p_2-1)}} \tag{40}$$

$$\left(resp.\ \ v(x) \leq c\ dist(x, \partial\Omega)^{-\frac{p_1 q_1 + p_2(p_1-1)}{q_1 q_2 - (p_1-1)(p_2-1)}} \right). \tag{41}$$

Remark 3.9 *The estimates proved in the above lemma above are sharp. That is, estimates (40) and (41) cannot be improved. To see this consider* $\Omega :=]0, +\infty[\times \mathbb{R}^{N-1}$. *In this case* $dist(x, \partial\Omega) = x_1$. *Let* $q_1 > p_1 - 1 > 0$ $q_2 > p_2 - 1 > 0$. *It is easy to see that the functions* u, v *defined by*

$$u(x) := x_1^{-\frac{p_2 q_2 + p_1(p_2-1)}{q_1 q_2 - (p_1-1)(p_2-1)}} \quad and \quad v(x) := x_1^{-\frac{p_1 q_1 + p_2(p_1-1)}{q_1 q_2 - (p_1-1)(p_2-1)}}$$

solve the system,

$$\begin{cases} \Delta_{p_1} u = \lambda v^{q_2} & \text{on } \Omega, \\ \Delta_{p_2} v = \mu u^{q_1} & \text{on } \Omega. \end{cases}$$

where $\lambda, \mu > 0$ *are suitable positive constants.*

Remark 3.10 *The above results, Lemma 3.1, 3.5, 3.6 3.7, 3.8, continue to hold (with the same proof as above) even in the case* $p_1 = 1$ *or* $p_2 = 1$.

4 Liouville Theorems

In this Section, if not otherwise specified, $\mathscr{A}, \mathscr{A}_1$ and $\mathscr{A}_2$ are **W**-p-**C** with indices $p > 1$, $p_1 > 1$ and $p_2 > 1$ respectively.

Our main result in this section is the following.

Theorem 4.1 *Let* (u, v) *be a weak solution of*

$$\begin{cases} \operatorname{div}_L \mathscr{A}_1(x, u, \nabla_L u) \geq v^{q_2} \quad on \quad \mathbb{R}^N, \\ \\ \operatorname{div}_L \mathscr{A}_2(x, v, \nabla_L v) \geq u^{q_1} \quad on \quad \mathbb{R}^N, \\ \\ u \geq 0, \quad v \geq 0 \quad on \quad \mathbb{R}^N. \end{cases} \tag{42}$$

If

$$\max\left\{q_1\frac{p_2q_2+p_1(p_2-1)}{q_1q_2-(p_1-1)(p_2-1)}+p_1,\ q_2\frac{p_1q_1+p_2(p_1-1)}{q_1q_2-(p_1-1)(p_2-1)}+p_2\right\}\geq Q, \tag{43}$$

then $u\equiv v\equiv 0$.

If $p_1=p_2=p$, *then (43) becomes*

$$\max\left\{q_1\frac{q_2+p-1}{q_1q_2-(p-1)^2},q_2\frac{q_1+p-1}{q_1q_2-(p-1)^2}\right\}\geq\frac{Q-p}{p},$$

which in turn, when $q_1=q_2=q$ *becomes*

$$q(Q-2p)\leq(Q-p)(p-1).$$

Remark 4.2 1. *Notice that (43) can be rewritten as*

$$\max\left\{p_1(\frac{1}{\delta_1}-1),p_2(\frac{1}{\delta_2}-1)\right\}\geq\frac{Q-p_1-p_2}{\delta_1\delta_2}-Q, \tag{44}$$

where $\delta_i:=\frac{p_i-1}{q_i}$, $i=1,2$.

2. *From (44) we easily see that if* $Q\leq p_1+p_2$ *then there are no nontrivial solution of (42) for any* q_1 *and* q_2 *such that* $q_i>p_i-1$, $i=1,2$.

Proof of Theorem 4.1. Define

$$f_0:=Q-q_1\frac{p_2q_2+p_1(p_2-1)}{q_1q_2-(p_1-1)(p_2-1)}\quad\textit{and}\quad g_0:=Q-q_2\frac{p_1q_1+p_2(p_1-1)}{q_1q_2-(p_1-1)(p_2-1)}.$$

Applying Lemma 3.1 to the first inequality of (42) with $s:=q_1-p_1+1$ we have

$$\int_{\mathbb{R}^N}v^{q_2}u^{q_1-p_1+1}\phi+c_1s\int_{\mathbb{R}^N}\mathscr{A}_1(x,u,\nabla_L u)\cdot\nabla_L u\ u^{q_1-p_1}\phi\leq c_2s^{1-p_1}\int_{\mathbb{R}^N}u^{q_1}\frac{|\nabla_L\phi|^{p_1}}{\phi^{p_1-1}}. \tag{45}$$

Taking $\phi=\phi_R$ a standard cut off function as in the proof of Lemma 3.7 and using the above estimates (35) we have

$$\int_{\mathbb{R}^N}u^{q_1}\frac{|\nabla_L\phi|^{p_1}}{\phi^{p_1-1}}\leq cR^{f_0-p_1}, \tag{46}$$

which, together with (45) implies

$$\int_{B_R}v^{q_2}u^{q_1-p_1+1}+c_1s\int_{B_R}\mathscr{A}_1(x,u,\nabla_L u)\cdot\nabla_L u\ u^{q_1-p_1}\leq cR^{f_0-p_1}. \tag{47}$$

Similarly we obtain that

$$\int_{B_R} u^{q_1} v^{q_2-p_2+1} + c\int_{B_R} \mathscr{A}_2(x,v,\nabla_L v)\cdot\nabla_L v\; v^{q_2-p_2} \le cR^{g_0-p_2}. \tag{48}$$

Since the hypothesis (43) can be written as $\min\{f_0-p_1, g_0-p_2\}\le 0$, we deduce that at least one of the exponent in (47) and (48) is nonpositive. Without loss of generality we assume that $f_0\le p_1$. Then, from (47) we obtain that

$$\mathscr{A}_1(x,u,\nabla_L u)\cdot\nabla_L u\; u^{q_1-p_1}\in L^1(\mathbb{R}^N). \tag{49}$$

Now arguing as in Lemma 3.1, using $u^{q_1-p_1+1}\phi$ as test function, we have

$$\int_{\mathbb{R}^N} v^{q_2}u^{q_1-p_1+1}\phi + (q_1-p_1+1)\int_{\mathbb{R}^N}\mathscr{A}_1(x,u,\nabla_L u)\cdot\nabla_L u\; u^{q_1-p_1}\phi \le \tag{50}$$

$$\le \int_S u^{q_1-p_1+1}\,|\mathscr{A}_1|\,|\nabla_L\phi| \tag{51}$$

$$\le \left(\int_S \mathscr{A}_1(x,u,\nabla_L u)\cdot\nabla_L u\; u^{q_1-p_1}\phi\right)^{1/p_1'}\left(\int_S u^{q_1}\frac{|\nabla_L\phi|_1^p}{\phi^{p_1-1}}\right)^{1/p_1} \tag{52}$$

$$\le \left(\int_S \mathscr{A}_1(x,u,\nabla_L u)\cdot\nabla_L u\; u^{q_1-p_1}\phi\right)^{1/p_1'} M, \tag{53}$$

where S is the support of $\nabla_L\phi$ and in the last inequality we have used (46) and the fact that $f_0-p_1\le 0$. Choosing again $\phi=\phi_R$ as above, from (53) we obtain

$$\int_{B_R}\mathscr{A}_1(x,u,\nabla_L u)\cdot\nabla_L u\; u^{q_1-p_1}\le c\left(\int_{B_{2R}\setminus B_R}\mathscr{A}_1(x,u,\nabla_L u)\cdot\nabla_L u\; u^{q_1-p_1}\right)^{1/p_1'},$$

which, with the information (49), by letting $R\to+\infty$, implies that

$$\mathscr{A}_1(x,u,\nabla_L u)\cdot\nabla_L u\; u^{q_1-p_1}\equiv 0 \quad \textit{on } \mathbb{R}^N.$$

From Remark 3.3 we have that $\mathscr{A}_1(x,u,\nabla_L u)\equiv 0$, and consequently, by the first inequality of (42), $v\equiv 0$. The same conclusion holds for u. □

Theorem 4.3 *Assume that either $\mathscr{A}_1$ or $\mathscr{A}_2$ satisfy the weak Harnack inequality (WH). If $q_i>p_i-1$, $i=1,2$, and (u,v) is a weak solution of (42), then $u\equiv v\equiv 0$.*

Proof of Theorem 4.3. Assume that for $\mathscr{A}_1$ the weak Harnack inequality holds. From estimate (35) we have

$$\sup_{B_{R/2}} u\le cR^{-\frac{p_2q_2+p_1(p_2-1)}{q_1q_2-(p_1-1)(p_2-1)}}.$$

By letting $R\to+\infty$ in the above inequality, the claim will follows. □

Corollary 4.4 *Assume that either* $\mathscr{A}_1$ *is* $\mathbf{S}$-p_1-$\mathbf{C}$ *or* $\mathscr{A}_2$ *is* $\mathbf{S}$-p_2-$\mathbf{C}$. *If* $q_i > p_i - 1$, $i = 1, 2$, *and* (u, v) *is a weak solution of (42), then* $u \equiv v \equiv 0$.

Proof. Assume that $\mathscr{A}_1$ is $\mathbf{S}$-p_1-$\mathbf{C}$. From Lemma 2.5 we have that for the nonnegative solutions of $\mathrm{div}_L(\mathscr{A}(x.u.\nabla_L u)) \geq 0$ the weak Harnack inequality holds. Thus Theorem 4.3 applies. □

In what follows we need of the following.

Lemma 4.5 *Let* $w \in L^1_{loc}(\mathbb{R}^N)$ *be nonnegative. For* $r > 0$ *define*[2]

$$m_w(r) := \operatorname{ess} \inf_{\nu(x)>r} w(x).$$

Let $f :]0, +\infty[\to]0, +\infty[$ *be a positive continuous nondecreasing function such that* $f(w) \in L^1_{loc}(\mathbb{R}^N)$. *If*

$$\liminf_{R} \fint_{B_R} f(w(x)dx = 0,$$

then $\lim_{t\to 0} f(t) = 0$ *and for any* $r > 0$, $m_w(r) = 0$.

Proof. Let $r_0 > 0$ and $R > r_0$. We have

$$\fint_{B_R} f(w) \geq \frac{1}{\omega_\nu R^Q} \int_{B_R \setminus B_{r_0}} f(w) \geq \frac{1}{R^Q}(R^Q - r_0^Q) f\left(m_u(r_0)\right).$$

Letting $R \to +\infty$ in the last chain of inequalities, we obtain $0 \geq f(m_u(r_0))$. □

Corollary 4.6 *Let* $q_i > p_i - 1$, $i = 1, 2$, *and let* (u, v) *be a weak solution of (42). Let* $r > 0$ *and set*

$$m_u(r) := \operatorname{ess} \inf_{\nu(x)>r} u(x), \qquad m_v(r) := \operatorname{ess} \inf_{\nu(x)>r} v(x).$$

Then for any $r > 0$ *we have* $m_u(r) = m_v(r) = 0$.

Proof. From estimates (34) and (35) it follows that

$$\left(\fint_{B_R} v^{q_2}\right)^{1/q_2} \quad \textit{and} \quad \left(\fint_{B_R} u^{q_1}\right)^{1/q_1}$$

vanish for $R \to +\infty$. By choosing $f(t) = t^{q_2}$ and $f(t) = t^{q_1}$ in Lemma 4.5, the claim follows. □

The next result deals with radial solution in the Euclidean framework. We need the following.

[2]We recall that $\nu(x)$ denotes the norm of x.

Definition 4.7 *We say that $\mathscr{A}$ is radial if there exists a Caratheodory function $A : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that for any $u \in \mathscr{C}^1(\mathbb{R}^N)$ radial, that is $u(x) = u(|x|)$, we have that*

$$\mathscr{A}(x, u, \nabla u) = A(|x|\,, u(|x|), u'(|x|))\frac{x}{|x|}.$$

Remark 4.8 *If $\mathscr{A}$ is radial and $u = u(|x|)$ is a radial solution of*

$$\operatorname{div}(\mathscr{A}(x, u, \nabla u) \geq f(|x|) \quad on \ \mathbb{R}^N,$$

then u solves

$$(r^{N-1}A(r, u(r), u'(r)))' \geq r^{N-1}f(r), \quad for \quad r > 0.$$

Theorem 4.9 *Let $\mathbb{G}$ be the Euclidean group $\mathbb{R}^N$. Let $q_i > p_i - 1$, $i = 1, 2$, and let (u, v) be a weak solution of (42). Assume that either $\mathscr{A}_1$ is radial and u is a $\mathscr{C}^1$ radial function or $\mathscr{A}_2$ is radial and v is a $\mathscr{C}^1$ radial function. Then $u \equiv v \equiv 0$.*

Proof. Assume that $\mathscr{A}_1$ is radial and u is a $\mathscr{C}^1$ radial function. Therefore u solves

$$(r^{N-1}A(r, u(r), u'(r)))' \geq 0 \quad for \quad r > 0.$$

Integrating between 0, and r we have $r^{N-1}A(r, u(r), u'(r)) \geq 0$ which together with the weakly ellipticity (WE) of $\mathscr{A}$ yields $u'(r) \geq 0$ for $r > 0$.

We proceed by contradiction assuming that $u \not\equiv 0$. Hence there exists $r_0 > 0$ such that $u(r_0) > 0$. Since u is nondecreasing we have that $u(r) \geq u(r_0) > 0$ for any $r > r_0$. This contradicts Corollary 4.6. $\square$

4.1 Nonautonomous systems

In this Section we briefly show how the ideas developed in the preceding Sections can be employed to study a class of nonautonomous systems.

We consider

$$\begin{cases} \operatorname{div}_L \mathscr{A}_1(x, u, \nabla_L u) \geq H(x)\, v^{q_2} \quad on \quad \mathbb{R}^N, \\ \\ \operatorname{div}_L \mathscr{A}_2(x, v, \nabla_L v) \geq K(x)\, u^{q_1} \quad on \quad \mathbb{R}^N, \\ \\ u \geq 0, \quad v \geq 0 \quad on \quad \mathbb{R}^N. \end{cases} \tag{54}$$

We assume that H and K are measurable functions such that for a.e. $x \in \mathbb{R}^N$

$$H(x) \geq \frac{c_H}{(1+\nu(x))^{\alpha_1}}, \qquad K(x) \geq \frac{c_K}{(1+\nu(x))^{\alpha_2}}, \tag{55}$$

where $\alpha_1, \alpha_2 \geq 0$ and $c_H, c_K > 0$.

We have the following.

Theorem 4.10 *Let $q_i > p_i - 1$, $i = 1, 2$ and let (u, v) be a weak solution of (54).*

1. *If*

$$\max\Bigg\{ q_1 \frac{(p_2-\alpha_2)q_2 + (p_1-\alpha_1)(p_2-1)}{q_1q_2 - (p_1-1)(p_2-1)} + p_1,\ q_1 \frac{(p_2-\alpha_2)q_2 + (p_1-\alpha_1)(p_2-1)}{q_1q_2 - (p_1-1)(p_2-1)} + \alpha_2,$$
$$q_2 \frac{(p_1-\alpha_1)q_1 + (p_2-\alpha_2)(p_1-1)}{q_1q_2 - (p_1-1)(p_2-1)} + p_2,\ q_2 \frac{(p_1-\alpha_1)q_1 + (p_2-\alpha_2)(p_1-1)}{q_1q_2 - (p_1-1)(p_2-1)} + \alpha_2 \Bigg\} > Q,$$

   then $u \equiv v \equiv 0$.

2. *Assume that either,*

$$\mathscr{A}_1 \text{ is } \mathbf{S} - p_1 - \mathbf{C} \quad \text{and} \quad (p_2-\alpha_2)q_2 + (p_1-\alpha_1)(p_2-1) > 0,$$

   or

$$\mathscr{A}_2 \text{ is } \mathbf{S} - p_2 - \mathbf{C} \quad \text{and} \quad (p_1-\alpha_1)q_1 + (p_2-\alpha_2)(p_1-1) > 0.$$

   Then $u \equiv v \equiv 0$.

Remark 4.11 *The above result complements and refines those obtained in [35]. Indeed, in [35] the author proved existence and nonexistence of radial solutions for a special systems of equations involving p and q-Laplacian operators.*

Since the proof of Theorem 4.10 follows the same ideas as in Theorem 4.1 and Theorem 4.3, we shall be brief. We first state some estimates which have an interest in themselves. These are the analogues of those obtained in Lemma 3.7, which in this case read as

$$\left(\fint_{B_R(x)} Hv^{q_2}\right)^{1/q_2} \leq cR^{-\frac{(p_1-\alpha_1)q_1+(p_2-\alpha_2)(p_1-1)}{q_1q_2-(p_1-1)(p_2-1)} - \frac{\alpha_1}{q_2}}, \tag{56}$$

$$\left(\fint_{B_R(x)} Ku^{q_1}\right)^{1/q_1} \leq cR^{-\frac{(p_2-\alpha_2)q_2+(p_1-\alpha_{)}(p_2-1)}{q_1q_2-(p_1-1)(p_2-1)} - \frac{\alpha_2}{q_1}}, \tag{57}$$

for any $x \in \mathbb{R}^N$ and $R > 0$. By using the hypotheses on H and K, it foolows that

$$\left(\fint_{B_R(x)} v^{q_2}\right)^{1/q_2} \leq cR^{-\frac{(p_1-\alpha_1)q_1+(p_2-\alpha_2)(p_1-1)}{q_1q_2-(p_1-1)(p_2-1)}}, \tag{58}$$

$$\left(\fint_{B_R(x)} u^{q_1}\right)^{1/q_1} \leq cR^{-\frac{(p_2-\alpha_2)q_2+(p_1-\alpha_{)}(p_2-1)}{q_1q_2-(p_1-1)(p_2-1)}}. \tag{59}$$

Inequalities (56), (57), (58), and (59), allow us to argue as in the proofs of Theorem 4.1 and Theorem 4.3 obtaining the claim.

Remark 4.12 *Notice that, the same kind of results hold if we replace the pointwise conditions (55) with the weaker ones*

$$\left(\fint_{R<\nu(x)<2R} H^{-\sigma}\right)^{1/\sigma} \leq C_H R^{\alpha_1}, \qquad \left(\fint_{R<\nu(x)<2R} K^{-\sigma}\right)^{1/\sigma} \leq C_K R^{\alpha_2},$$

for R large and suitable $\sigma > 0$. These kind of integral hypotheses on the coefficients have already been used in [11] and [12]. We refer the interested reader to those works for further details.

5 Systems with changing sign nonlinearities

Let $Lw = \mathrm{div}_L \mathscr{A}(x, w, \nabla_L w)$ be as above. Let (u, v) be a solution of

$$\begin{cases} L(u) \geq f(v) & \text{on } \Omega, \\ L(v) \geq g(u) & \text{on } \Omega. \end{cases} \tag{60}$$

Clearly the pair (u, v) solves the inequality

$$L(u) + L(v) \geq g(u) + f(v) \qquad \text{on } \Omega. \tag{61}$$

In particular if L is odd (that is $L(-u) = -L(u)$), then setting $w := -u$ and $\tilde{g}(t) := -g(-t)$, the couple $(v, w) = (v, -u)$ solves the inequality

$$L(v) - f(v) \geq L(w) - \tilde{g}(w) \qquad \text{on } \Omega. \tag{62}$$

In order to study (62) we need of the following.

Definition 5.1 *We say that $\mathscr{A}\colon \mathbb{R}^N \times \mathbb{R}^l \to \mathbb{R}^l$ is monotone if*

$$(\mathscr{A}(x, \xi) - \mathscr{A}(x, \eta)) \cdot (\xi - \eta) \geq 0 \qquad for \quad x \in \mathbb{R}^N,\ \xi, \eta \in \mathbb{R}^l. \tag{63}$$

We say that $\mathscr{A}$ is **M**-p-**C** *(monotone p-coercive) if $\mathscr{A}$ is monotone and there exists $k > 0$ such that*

$$(\mathscr{A}(x, \xi) - \mathscr{A}(x, \eta) \cdot (\xi - \eta))^{p-1} \geq k^{p-1} \left|\mathscr{A}(x, \xi) - \mathscr{A}(x, \eta)\right|^p \qquad for \quad x \in \mathbb{R}^N,\ \xi, \eta \in \mathbb{R}^l. \tag{64}$$

Examples of **M**-p-**C** operators can be found for instance in D'Ambrosio, Farina and Mitidieri [10]. Among others we have,

Example 5.2 *1. Let $1 < p \leq 2$, then the function $\mathscr{A}(\xi) := |\xi|^{p-2}\xi$ generates an* **M**-p-**C** *operator.*

2. The mean curvature operator is $\mathbf{M}$-p-$\mathbf{C}$ *for* $1 < p \le 2$.

In [10] the authors prove the following comparison result.

Theorem 5.3 *Let* $\mathscr{A}$ *be* $\mathbf{M}$-p-$\mathbf{C}$ *and* $q > \min\{1, p-1\}$. *Let (u,v) be a weak solution of*

$$\operatorname{div}_L\left(\mathscr{A}(x, v, \nabla_L v)\right) - |v|^{q-1}\, v \ge \operatorname{div}_L\left(\mathscr{A}(x, u, \nabla_L u)\right) - |u|^{q-1}\, u \qquad \text{on } \mathbb{R}^N. \tag{65}$$

Then $v \le u$ *a.e. on* $\mathbb{R}^N$.

Remark 5.4 *Notice that the above claim* $v \le u$ *a.e. on* $\mathbb{R}^N$, *is fairly easy to prove under locally bounded assumption of the weak solutions. For further details on this point see [10].*

As consequence of Theorem 5.3, we prove the following.

Theorem 5.5 *Let* $\mathscr{A}$ *be* $\mathbf{M}$-p-$\mathbf{C}$, $q > \min\{1, p-1\}$ *and let* (u, v) *be a weak solution of*

$$\begin{cases} \operatorname{div}_L\left(\mathscr{A}(x, \nabla_L u)\right) \ge |v|^{q-1}\, v & \text{on } \mathbb{R}^N, \\ \\ \operatorname{div}_L\left(\mathscr{A}(x, \nabla_L v)\right) \ge |u|^{q-1}\, u & \text{on } \mathbb{R}^N. \end{cases} \tag{66}$$

Then $u + v \le 0$ *a.e. on* $\mathbb{R}^N$.

Moreover, if $\mathscr{A}$ *is odd, that is* $\mathscr{A}(x, -\xi) = -\mathscr{A}(x, \xi)$, *and* (u, v) *solves also the equation in (66), then* $u = -v$ *a.e. on* $\mathbb{R}^N$ *and* u *solves*

$$-\operatorname{div}_L\left(\mathscr{A}(x, \nabla_L u)\right) = |u|^{q-1}\, u \qquad \text{on } \ \mathbb{R}^N.$$

Proof. Arguing as at the beginning of this section, $(-u, v)$ is a solution of (65). Hence by Theorem 5.3 it follows that $v \le -u$. This completes the first part of the proof.

Now, if (u, v) is a solution of (66) with equality sign, then $(-u, -v)$ solves the same equations. By the first part of this claim we deduce that $-u - v \le 0$, thereby concluding the proof. □

Corollary 5.6 *Let* $\mathscr{A}$ *be* $\mathbf{M}$-p-$\mathbf{C}$ *and odd. Let* $q > \min\{1, p-1\}$ *and let* (u, v) *be a weak solution of*

$$\begin{cases} -\operatorname{div}_L\left(\mathscr{A}(x, \nabla_L u)\right) = |v|^{q-1}\, v & \text{on } \mathbb{R}^N, \\ \\ -\operatorname{div}_L\left(\mathscr{A}(x, \nabla_L v)\right) = |u|^{q-1}\, u & \text{on } \mathbb{R}^N. \end{cases} \tag{67}$$

Then $u \equiv v$ *a.e. on* $\mathbb{R}^N$.

Proof. The claim follows by observing that $(-u, v)$ solves the system (66) with equality signs. Hence the claim follows from Theorem 5.5. □

Corollary 5.7 *Let* $\mathscr{A}$ *be* $\mathbf{M}$-p-$\mathbf{C}$, $q > \min\{1, p-1\}$ *and let* (u, v) *be nonnegative weak solution of (66), then* $u \equiv v \equiv 0$.

Some open questions

Obviously, there are many interesting questions one may ask on the problems considered in this paper. We find the following particularly intriguing.

1. We proved our main result under the restrictive hypothesis

$$q_1 > p_1 - 1,$$

and

$$q_2 > p_2 - 1.$$

The main question is that if the same result as in Theorem 4.1 holds under the weaker natural assumption (see [1])

$$q_1 q_2 > (p_1 - 1)(p_2 - 1).$$

Clearly, this problem is meaningful if one knows that Harnack's inequality does not hold.

2. Is the curve in Theorem 4.1 sharp? We believe that the answer is no.

3. If the nonlinearities are not power functions, is there any condition *a la* Keller-Osserman implying that the only solution of the system (5) is the trivial one?

4. Does the Liouville's theorem hold if in system (8) we replace the power functions with increasing nonlinearities?

5. What about Theorem 1.4 when the power of the nonlinearity is not the same?

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References

[1] M.F. Bidaut-Véron, M. Garcia-Huidobro and C. Yarur, Keller-Osserman estimates for some quasilinear elliptic systems, `arXiv:1102.2564`, (2011).

[2] M.F. Bidaut-Véron and S.I. Pohozaev, Nonexistence results and estimates for some nonlinear elliptic problems, *J. Anal. Math.* **84** (2001) 1–49.

[3] A. Bonfiglioli, E. Lanconelli and F. Uguzzoni, Stratified Lie Groups and Potential Theory for their sub-Laplacians, Springer Monogr. Math., 26. New York, NY: Springer-Verlag, 2007.

[4] J. Busca and R. Manasevich, A Liouville-type theorem for Lane-Emden system, *Indiana Univ. Math. J.* **51** (2002) 37–51.

[5] L. Capogna, D. Danielli and N. Garofalo, Embedding theorem and the Harnack inequality for solutions of nonlinear subelliptic equations, *Comm. Partial Differential Equations* **18** (1993) 1765–1794.

[6] G. Caristi, L. D'Ambrosio and E. Mitidieri, Representation Formulae for Solutions to Some Classes of Higher Order Systems and Related Liouville Theorems, *Milan J. Math* **76** (2008) 27–67.

[7] F.-C.St. Cîrstea and V. Rădulescu, Entire solutions blowing up at infinity for semilinear elliptic systems, *J. Math. Pures Appl.* **81** (2002) 827–846.

[8] F. Colasuonno, L. D'Ambrosio and E. Mitidieri, Entire solutions of quasilinear elliptic systems with general nonlinearities on $\mathbb{R}^N$, (2012) preprint.

[9] L. D'Ambrosio, Liouville theorems for anisotropic quasilinear inequalities *Nonlinear Anal.* **70** (2009) 2855–2869.

[10] L. D'Ambrosio, A. Farina and E. Mitidieri, Comparison principles, uniqueness and simmetry results of solutions of quasilinear elliptic equations and inequalities, (2012) preprint.

[11] L. D'Ambrosio and E. Mitidieri, A priori estimates, positivity results, and nonexistence theorems for quasilinear degenerate elliptic inequalities, *Adv. Math.* **224** (2010) 967–1020.

[12] L. D'Ambrosio and E. Mitidieri, A Priori Estimates and Reduction Principles for Quasilinear Elliptic Problems and Applications, *Adv. Differential Equations* **17** (2012) 935–1000.

[13] L. D'Ambrosio and E. Mitidieri, Entire solutions of quasilinear elliptic systems on Carnot Groups, *Proc. Steklov Inst. Math.* **283** (2013) to appear.

[14] L. D'Ambrosio, E. Mitidieri and S.I. Pohozaev, Representation Formulae and Inequalities for Solutions of a Class of Second Order Partial Differential Equations, *Trans. Amer. Math. Soc.*, **358** (2006) 893–910.

[15] A. Farina and J. Serrin, Entire solutions of completely coercive quasilinear elliptic equations, *J. Differential Equations* **250** (2011) 4367–4408.

[16] A. Farina and J. Serrin, Entire solutions of completely coercive quasilinear elliptic equations II, *J. Differential Equations* **250** (2011) 4409–4436.

[17] R. Filippucci, Nonexistence of positive weak solutions of elliptic systems of divergence type, *J. Differential Equations* **250** (2011) 572-595.

[18] R. Filippucci, On entire solutions of quasilinear elliptic systems with gradient terms, (2012) preprint.

[19] G.B. Folland, Subelliptic estimates and function spaces on nilpotent Lie groups, *Ark. Mat.* **13** (1975) 161–207.

[20] G.B. Folland and E.M. Stein, Hardy spaces on homogeneous groups, Math. Notes, vol. 28, Princeton University Press, Princeton, NJ, 1982.

[21] A.V. Lair and A.W. Wood, Existence of Entire Large Positive Solutions of Semilinear Elliptic Systems, *J. Differential Equations* **164** (2000) 380–394.

[22] J. Maly and W.P. Ziemer, Fine Regularity of Solutions of Elliptic Differential Equations, Mathematical Surveys and Monographs, AMS 1997.

[23] E. Mitidieri, A Rellich type identity and applications, *Comm. Partial Differential Equations* **18** (1993) 125–151.

[24] E. Mitidieri, Non existence of positive solutions of semilinear elliptic systems in $\mathbb{R}^{\mathbb{N}}$, *Differential Integral Equations* **9** (1996) 465–479.

[25] E. Mitidieri and S.I. Pohozaev, Nonexistence of Positive Solutions for a Systems of Quasilinear Elliptic Equations and Inequalities in $\mathbb{R}^N$, *Dokl. Math.* **59** (1999) 351–355.

[26] E. Mitidieri and S.I. Pohozaev, Non Existence of Positive Solutions for Quasilinear Elliptic Problems on $\mathbb{R}^{\mathbb{N}}$, *Proc. Steklov Inst. Math.* **227** (1999) 186-216.

[27] E. Mitidieri and S.I. Pohozaev, A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities, *Proc. Steklov Inst. Math.* **234** (2001) 1–362.

[28] Q.P. Phan, Liouville-type theorems and bounds of solutions for Hardy-Hénon elliptic systems, *Adv. Differential Equations* **17** (2012) 605–634.

[29] P. Polacik, P. Quittner and Ph. Souplet, Singularity and decay estimates in superlinear problems via Liouville-type theorems. Part I: Elliptic systems, *Duke Math. J.* **139** (2007) 555–579.

[30] P. Pucci and J. Serrin, The Strong Maximum Principle, *"Progress in Nonlinear Differential Equations and their Applications"*, **73**, Birkhäuser Publ., Switzerland, 2007.

[31] J. Serrin, Local behavior of solutions of quasilinear equations, *Acta Math.* **111** (1964) 247–302.

[32] J. Serrin and H. Zou, Non-existence of positive solutions of Lane-Emden systems *Differential Integral Equations* **9** (1996), 635–653.

[33] P. Souplet, The proof of the Lane-Emden conjecture in four space dimensions, *Adv. Math.* **221** (2009) 1409–1427.

[34] E.M. Stein and G. Weiss, Fractional Integrals in n-dimensional Euclidean space, *J. Math. Mech.* **7** (1958) 503–514.

[35] T. Teramoto, Existence and nonexistence of positive radial entire solutions of second order quasilinear elliptic systems, *Hiroshima Math. J.* **30** (2000) 437–461

[36] N.S. Trudinger, On Harnack type inequalities and their application to quasilinear elliptic equations, *Comm. Pure Appl. Math.* **20** (1967) 721–747.

[37] N.S. Trudinger and X.J. Wang, On the weak continuity of elliptic operators and applications to potential theory, *Amer. J. Math.* **124** (2002) 369–410.

[38] Z. Yang, Existence of entire explosive positive radial solutions for a class of quasilinear elliptic systems, *J. Math. Anal. Appl.* **288** (2003) 768–783.

[39] C.S. Yarur, Nonexistence of Positive Singular Solutions for a Class of Semilinear Elliptic Systems, *Electron. J. Differential Equations* **1996** (1996) 1–22.